定理 6.4:

Pf: (1) 先证 $\partial A = \bar{A} \cap \overline{A^c}$

若 $x \in \partial A$. 由边界点定义得, $\forall r>0$, $B(x;r) \cap A \neq \phi$, $B(x;r) \cap A^c \neq \phi$

由定理 5.4 知, $x \in \bar{A}$, $x \in \overline{A^c}$ $\Rightarrow$ $x \in \bar{A} \cap \overline{A^c}$

若 $x \in \bar{A} \cap \overline{A^c}$, 同理可得, $\forall r>0$, $B(x;r) \cap A \neq \phi$, $B(x;r) \cap A^c \neq \phi$

$\therefore$ $x \in \partial A$

又由定理 6.1 知: $\bar{A} \cap \overline{A^c} = ((A^c)^o)^c \cap (A^o)^c$

$= ((A^c)^o \sqcup A^o)^c$ (De Morgan 律 $(A \cap B)^c = A^c \sqcup B^c$)

显然有 $\partial A = \bar{A} \cap \overline{A^c} = \partial A^c$.

(2). $A^o \sqcup \partial A = A^o \sqcup (\bar{A} \cap \overline{A^c})$

$= (A^o \sqcup \bar{A}) \cap (A^o \sqcup \overline{A^c})$ [$\because$ $A^o \subset A \subset \bar{A}$, $\overline{A^c} = (A^o)^c$ (定理 6.1)

$= \bar{A} \cap (A^o \sqcup (A^o)^c) = \bar{A}$.

$A^o \cap \partial A = \phi$ [$\because$ 若 $\exists x \in A^o \cap \partial A$, 则 由 $x \in A^o$, $\exists r>0$, s.t. $B(x;r) \subset A$.

则 $B(x;r) \cap A^c = \phi$ 与 $x \in \partial A$ 矛盾)

$\therefore$ $\bar{A} = A^o \sqcup \partial A$ (不交并)

显然有 $A^o = \bar{A} \setminus \partial A$, $\partial A = \bar{A} \setminus A^o$.

(3). $A \sqcup \partial A = A \sqcup (\bar{A} \cap \overline{A^c})$

$= (A \sqcup \bar{A}) \cap (A \sqcup \overline{A^c})$ [$A \subset \bar{A}$; $\mathbb{R}^n = A \sqcup A^c \subset A \sqcup \overline{A^c}$.]

$= \bar{A} \cap \mathbb{R}^n = \bar{A}$

$A - \partial A = A - (\bar{A} \cap \overline{A^c}) = A \cap ((\bar{A})^c \sqcup (\overline{A^c})^c)$ [$\because$ $A^o = (\overline{A^c})^c$]

$= (A \cap (\bar{A})^c) \sqcup (A \cap A^o)$ [$A \subset \bar{A} \Rightarrow (\bar{A})^c \subset A^c$; $A^o \subset A$]

$A \cap (\bar{A})^c \subset A \cap A^c = \phi$

$= \phi \sqcup A^o = A^o$

(4). $\partial A^o = \overline{A^o} \cap \overline{(A^o)^c} \subset \bar{A} \cap \overline{(A^o)^c} = \bar{A} \cap \overline{A^c} = \partial A$

[$\because$ $A^o = (\overline{A^c})^c$ $\Rightarrow$ $(A^o)^c = \overline{A^c}$ $\Rightarrow$ $\overline{(A^o)^c} = \overline{\overline{A^c}} = \overline{A^c}$.

$\partial \bar{A} = \bar{\bar{A}} \cap \overline{(\bar{A})^c} = \bar{A} \cap \overline{(\bar{A})^c} \subset \bar{A} \cap \overline{A^c} = \partial A$.

$[\because A \subset \overline{A} \Rightarrow (\overline{A})^c \subset A^c \Rightarrow \overline{(\overline{A})^c} \subset \overline{A^c}]$

(5) $\partial(A \cup B) = \overline{A \cup B} \cap \overline{(A \cup B)^c}$

$= (\overline{A} \cup \overline{B}) \cap \overline{(A \cup B)^c}$

$= (\overline{A} \cap \overline{(A \cup B)^c}) \cup (\overline{B} \cap \overline{(A \cup B)^c})$

$[\because A \subset A \cup B \Rightarrow (A \cup B)^c \subset A^c \Rightarrow \overline{(A \cup B)^c} \subset \overline{A^c} \Rightarrow \overline{A} \cap \overline{(A \cup B)^c} \subset \overline{A} \cap \overline{A^c}$

同理，$\overline{B} \cap \overline{(A \cup B)^c} \subset \overline{B} \cap \overline{B^c}$

$\subset (\overline{A} \cap \overline{A^c}) \cup (\overline{B} \cap \overline{B^c}) = \partial A \cup \partial B.$

(6) $\because \partial A = \overline{A} \cap \overline{A^c}$ 为闭集

$\therefore \overline{\partial A} = \partial A$

$\therefore \partial(\partial A) = \overline{\partial A} \cap \overline{(\partial A)^c} \subset \overline{\partial A} = \partial A.$

(7). 若 $\partial A = \emptyset$ 则由(3)知 $\overline{A} = A$，$A^\circ = A$

$\therefore A$既闭又开

若 A 既闭又开，则 $A = \overline{A}$，$A = A^\circ \Rightarrow (A^\circ)^c = A^c \overset{\text{定理6.1}}{\Rightarrow} \overline{A^c} = A^c$

$\therefore \partial A = \overline{A} \cap \overline{A^c} = A \cap A^c = \emptyset$

(8) ① 若 A 为闭集，$\overline{A} = A$

则 $\partial A = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \subset A.$

若 $\partial A \subset A$ 则由(3)知

$\overline{A} = A \cup \partial A = A \Rightarrow A$ 为闭

② 若 A 为开，$A = A^\circ$

则 $A \cap \partial A = \underline{A \cap \overline{A}} \cap \overline{A^c} = A \cap \overline{A^c} = A^\circ \cap \overline{A^c} \overset{\text{定理6.1}}{=} (\overline{A^c})^c \cap \overline{A^c} = \emptyset$

反之，若 $A \cap \partial A = \emptyset$

即 $\emptyset = A \cap \partial A = A \cap \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \overset{\text{定理6.1}}{=} A \cap (A^\circ)^c$

$\therefore A \subset ((A^\circ)^c)^c = A^\circ$

又 $A^\circ \subset A$

$\Rightarrow A = A^\circ$ A为开